11 april 2014

Topic:

MHD Flux Theorem & Energy Principle

Outline:

I. Review & Preview: Quick overview of how this week's topics are related to others, such as:

A. MHD equations (29 Nov 2013)

B. MHD waves (24 Jan 2014)

C. MHD equilibria (20 Dec 2013)

- magnetic pressure & tension
- θ-pinch
- z-pinch
- screw pinch
- force-free

(problem set from 20 Dec.)

D. MHD instabilities (coming next = 25 April 2014)

- apply energy principle to plasma/vacuum/wall system $\longrightarrow$ "intuitive" form
- "good" and "bad" curvature
- stability of pinches
- naming, classification, modes

E. MHD equilibria in toroidal configurations (16 May '14)

- toroidal force balance
- stability (surface current model)

II. Frozen-in flux:
- derivation
- examples
- implications

III. Energy principle:
- mechanical example
- derivation: $\rightarrow$ define equilibrium & linearize MHD eq.
  $\rightarrow$ define perturbation
  $\rightarrow$ conservation of energy + self-adjointness of $\underline{F}$ $\Rightarrow$ expression for δW
- what is it good for?

I. Review & Preview:

A. MHD equations

- They describe the plasma in terms of macroscopic properties of a single, conducting fluid.
- They don't describe all plasmas or all phenomena within the same plasma, but they are essential to the "big picture" — and, furthermore, work surprisingly well surprisingly often.

continuity: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$

equation of motion: (a.k.a., fluid eq.)
$$\underbrace{\rho\left(\frac{\partial u}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u}\right)}_{\frac{Du}{Dt} = \text{"convective derivative"}} = j \times \underline{B} - \underline{\nabla} p \ ^{\circledast}$$

equation of state: (a.k.a., energy eq.)
$$\frac{p^{\circledast}}{\rho^{\gamma}} = \text{constant}$$
(i.e., adiabatic relation; $\gamma = 5/3$ for 3-D systems)

$\circledast$ scalar p indicates assumption of isotropic pressure ($p_\perp = p_\parallel$)

Faraday's Law: $\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$

Ampere's Law: $\underline{\nabla} \times \underline{B} = \mu_0 j$

Ohm's Law: $\underline{E} + \underline{u} \times \underline{B} = \eta j \xrightarrow[\eta \to 0]{\lim} 0$ "ideal"

NOTE 1: The curl of Ohm's Law is also known as the "induction eq."

NOTE 2: The "generalized Ohm's Law" has 2 more terms on RHS:

a.) Hall term $\frac{1}{ne} j \times \underline{B}$ ← assumed negligibly small

b.) e^- pressure term $\frac{1}{en} \nabla p_e$ ← absent from induction eq; may also be negligibly small

MHD can be derived from 2-fluid OR guiding center theory.

★ Assumptions: $L \gg \lambda_D$, $\underline{u} \ll c$, $\underline{\nabla} p \parallel \underline{\nabla} n$, $\tau \gg \frac{1}{f_{ic}}, \frac{1}{f_{ec}}$

"Play" with the MHD equations to answer:

1.) What sorts of equilibria are possible?
2.) Are those equilibria stable?
3.) How can the system oscillate?
4.) What other interesting features/constants are possible? (frozen-in flux, helicity)

B. MHD Waves

- Idea: Even in a really boring MHD system ($\underline{B}$ the same everywhere, no current or flows, uniform pressure), there are different types of waves that can propagate.
  → We want to know their characteristics.

- Approach: 1.) Linearize equations ← Also used for energy principle, but now w/ a more interesting equilibrium.

  2.) Simplified Fourier analysis

As a result, approach to finding oscillation freq. & growth rate changes. (No more assuming a plane wave.)

C. MHD Equilibria: $\underline{j} \times \underline{B} = \underline{\nabla} p$

- flux surfaces: $\underline{B} \cdot \underline{\nabla} p = 0$
- current surfaces: $\underline{j} \cdot \underline{\nabla} p = 0$

That is, $\underline{j}$ and $\underline{B}$ are both ⊥ to $\underline{\nabla} p$, and the angle between them determines the size of the pressure gradient.

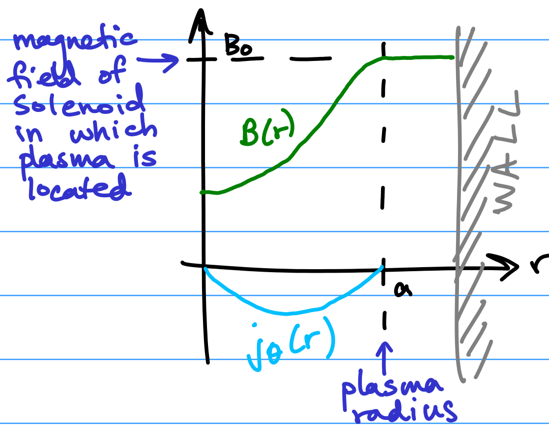
- equilibrium condition rewritten: (Use Ampère's Law to eliminate $\underline{j}$.)

$$\underline{\nabla}\Big(\underbrace{p}_{\text{pressure}} + \underbrace{\frac{B^2}{2\mu_0}}_{\substack{\text{magnetic "pressure"} \\ \text{(energy density)}}}\Big) = \underbrace{\frac{1}{\mu_0}(\underline{B}\cdot\underline{\nabla})\underline{B}}_{\text{magnetic "tension"}}$$

- examples of solutions in cylindrical coordinates:

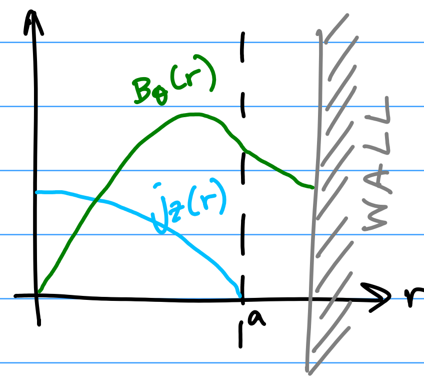
→ θ-pinch: $\underline{j} = j_\theta(r)\,\hat{\theta}$

$\underline{B} = B_z(r)\,\hat{z}$



→ z-pinch: $\underline{j} = j_z(r)\,\hat{z}$

$\underline{B} = B_\theta(r)\,\hat{\theta}$



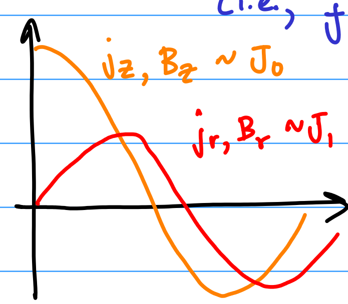
→ screw pinch: $\underline{j}$ & $\underline{B}$ both helical

These all produce the desired pressure profile for confinement:

- p peaked on axis
- p > 0 everywhere

→ force-free: $\underline{j} \times \underline{B} = \nabla p = 0$ ← No good for confinement, but found in nature.

(i.e., $\underline{j} \parallel \underline{B}$)



(Solutions are Bessel functions of the first kind.)

D & E = Future topics in MHD equilibrium & stability:

⇒ Use what we learn today about frozen-in flux and the energy principle to address questions such as:

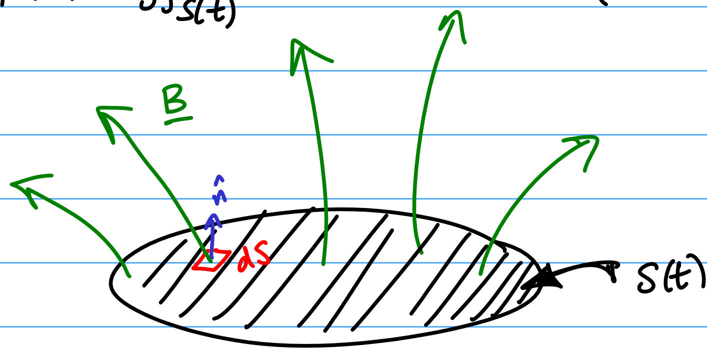
- Are those cylindrical equilibria stable?
- If not, what do the instabilities look like?
- What if you then bend the cylinder into a torus? Can you find a stable equilibrium then?
- Are there features that typically characterize a stable MHD equilibrium?
- If so, what are they?
- What about oscillations &/or instability growth rate?

Q: Why spend so much time on MHD stability, equilibrium, & features, when we know (e.g., from studying waves) that that isn't the whole picture?

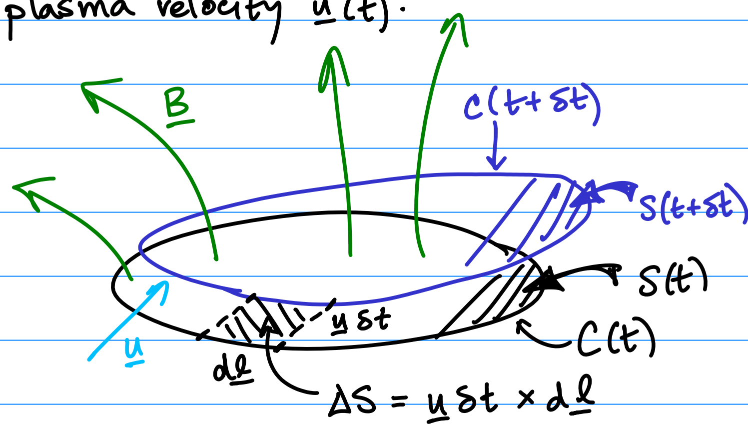
A: MHD stability is fundamental for magnetic fusion reactors. If your proposed reactor isn't even MHD stable, that means your entire conducting fluid is liable to move rapidly & coherently toward the wall, resulting in loss of the plasma and potentially even damage to the wall.

II. Frozen-in flux:

magnetic flux: $\phi(t) = \iint_{S(t)} \underline{B}(\underline{x}, t) \cdot d\underline{s}$ (where $d\underline{s} \equiv \hat{n}\, dS$)



change in magnetic flux through surface $S(t)$ bounded by contour $C(t)$ moving with plasma velocity $\underline{u}(t)$:



$$\frac{D\phi}{Dt} = \lim_{\delta t \to 0} \left[\frac{\iint_{S(t+\delta t)} \underline{B}(\underline{x}, t+\delta t) \cdot d\underline{s} - \iint_{S(t)} \underline{B}(\underline{x}, t) \cdot d\underline{s}}{\delta t} \right]$$

$$= \lim_{\delta t \to 0} \left[\frac{\iint_{S(t+\delta t)} \left(\underline{B} + \delta t \frac{\partial \underline{B}}{\partial t} \right) \cdot d\underline{s} - \iint_{S(t)} \underline{B} \cdot d\underline{s}}{\delta t} \right]$$

$$= \lim_{\delta t \to 0} \left[\frac{\underbrace{\iint_{S(t)} \left(\underline{B} + \delta t \frac{\partial \underline{B}}{\partial t} \right) \cdot d\underline{s}}_{\text{change in flux due to change in } \underline{B}} + \underbrace{\oint_{C(t)} \underline{B} \cdot \underline{u}\, \delta t \times d\underline{l}}_{\text{change in flux due to movement of contour}} \Big) - \iint_{S(t)} \underline{B} \cdot d\underline{s}}{\delta t} \right]$$

NOTE: Integrand of 2nd term could also be written $\underline{B} \times \underline{u}\, \delta t \cdot d\underline{l}$ (via vector identity).

$$= \lim_{\delta t \to 0} \left[\frac{\iint_{S(t)} \left(\underline{B} + \delta t \frac{\partial \underline{B}}{\partial t} \right) \cdot d\underline{s} + \oint_{C(t)} \underline{B} \times \underline{u}\, \delta t \cdot d\underline{\ell} \Big) - \iint_{S(t)} \underline{B} \cdot d\underline{s}}{\delta t} \right]$$

First, these cancel.

Next, remaining terms all have δt that cancels.
(And term in brackets is constant as $\delta t \to 0$.)

$$= \iint_{S(t)} \frac{\partial \underline{B}}{\partial t} \cdot d\underline{s} + \oint_{C(t)} \underline{B} \times \underline{u} \cdot d\underline{\ell}$$

$$= \iint_{S(t)} \underbrace{\left(\frac{\partial \underline{B}}{\partial t} + \underline{\nabla} \times (\underline{B} \times \underline{u}) \right)} \cdot d\underline{s} \qquad \text{(via Stokes's Theorem)}$$

Compare to curl of ideal Ohm's law: $\underline{\nabla} \times \left(\underline{E} + \underline{u} \times \underline{B} = 0 \right)$

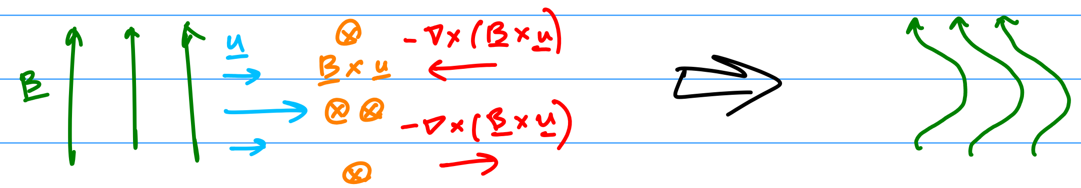
$$-\frac{\partial \underline{B}}{\partial t} + \underline{\nabla} \times (\underline{u} \times \underline{B}) = 0$$

$$\frac{\partial \underline{B}}{\partial t} + \underline{\nabla} \times (\underline{B} \times \underline{u}) = 0$$

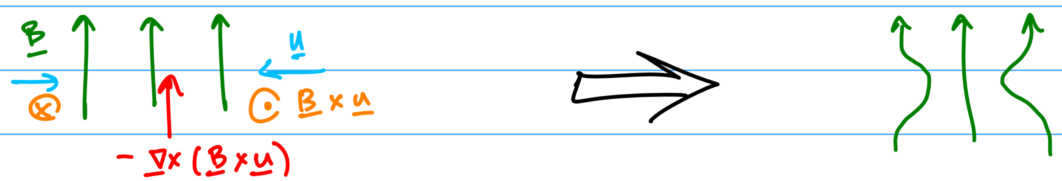
$\therefore$ If $\underline{E} + \underline{u} \times \underline{B} = 0$ (i.e., ηj negligible), then $\frac{D\phi}{Dt} \equiv \left(\frac{\partial}{\partial t} + \underline{u} \cdot \underline{\nabla} \right) \phi = 0$

$\Rightarrow$ In the frame of the moving plasma, magnetic flux is constant. This is the "frozen-in flux" concept.

Example A: (bending)



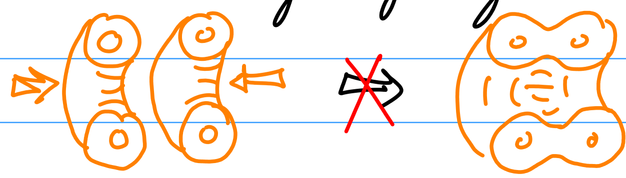
Example B: (compression)



Note 1: Frozen-in flux is NOT always the same as frozen-in magnetic field lines. An ideal MHD plasma can still $\underline{E} \times \underline{B}$ drift across magnetic field lines so long as the magnetic flux through every volume element is still preserved.

Note 2: Frozen-in flux is related to stability in that it:
- imposes a strong constraint on types of instabilities that can develop in an ideal MHD system,
- offers insight into how those instabilities might progress,
- and, importantly, how to stabilize them.

Note 3: Frozen-in flux means that an ideal MHD plasma can never undergo topological changes.

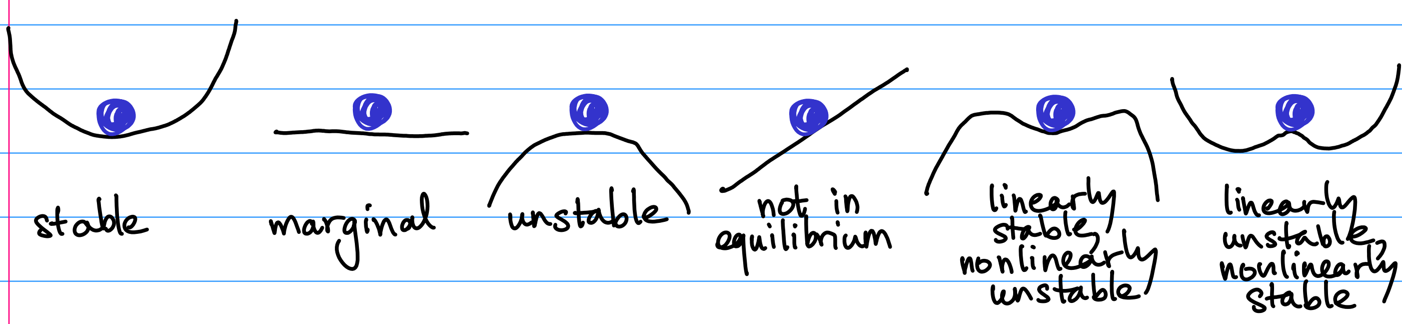


E.g., two tori being pushed together will never merge into one

The study of how plasmas that are usually well described by ideal MHD nevertheless undergo such changes (and do so much more rapidly than one would expect from simply using the resistive form of Ohm's Law) is a very active research area in plasma physics known as magnetic reconnection.

III. MHD energy principle

Equilibria (or not) illustrated:

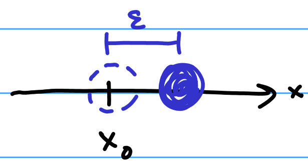


δW = change in potential energy when system is perturbed

$= -\delta K$ (change in kinetic energy) in a conservative system

(where $W + K = \text{constant}$)

Start by considering the example of a 1-D conservative system (such as various mechanical systems):



Assume:

- initial position: $x|_{t=0} = x_0$
- start from equilibrium:
$$F(x_0) = -\frac{dW}{dx}\bigg|_{x=x_0} = 0$$
- small displacement $\varepsilon(t)$

⇒ We want to know how this perturbation evolves in time.

Solve:

Newton's 2nd Law: $m\ddot{x} = F(x) = -\dfrac{dW}{dx}$

$x(t) = x_0 + \varepsilon(t)$

$\ddot{x}(t) = \ddot{\varepsilon}(t)$

Linearize (b/c ε is small):

$$\frac{dW}{dx}(x_0+\varepsilon) = \cancel{\frac{dW}{dx}\bigg|_{x_0}} + \varepsilon \frac{d^2W}{dx^2}\bigg|_{x_0} + \mathcal{O}(\varepsilon^2)$$

0 (since x_0 is an equilibrium point)

$$m\ddot{\varepsilon} = -\frac{d^2W}{dx^2}\bigg|_{x_0}\varepsilon$$

Therefore, if
$$\begin{cases} \dfrac{d^2W}{dx^2} > 0 \rightarrow \varepsilon(t) \text{ oscillates sinusoidally} \\ \dfrac{d^2W}{dx^2} < 0 \rightarrow \varepsilon(t) \text{ grows exponentially} \end{cases}$$

Introduce "δW" by expanding $W(x)$ around x_0:

$$W(x) = W(x_0) + \varepsilon\cancel{\frac{dW}{dx}\bigg|_{x_0}} + \frac{\varepsilon^2}{2}\frac{d^2W}{dx^2}\bigg|_{x_0} + \mathcal{O}(\varepsilon^3)$$

0

$$\equiv W_0 + \delta W$$

second order

$\delta W > 0 \leftrightarrow$ linearly stable

$\delta W < 0 \leftrightarrow$ linearly unstable

Now, we want to do something similar for an MHD system, which is a bit more complex.

→ effect of perturbation must be consistent with the system's governing equations & boundary conditions

Game plan:

1.) linearize MHD equations around a stationary, no-flow equilibrium

2.) define the perturbation $\underline{\xi}(\underline{x}, t)$ and combine with linearized MHD to get

$$\rho_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = \underline{F}(\underline{\xi})$$

(compare to $m\ddot{\varepsilon}$ in mechanical example)

3.) use energy conservation & self-adjointness of $\underline{F}$ to get expression for δW

Step 1:

all zeroth order terms independent of time

$\underline{u}_0 = 0$

- For a stationary, no flow MHD equilibrium:

$$\rho_0 = \rho_0(\underline{x})$$
$$\underline{u}_0 = 0$$
$$\underline{E}_0 = 0$$
$$\underline{\nabla} \cdot \underline{B}_0 = 0$$
$$\underline{\nabla} \times \underline{B}_0 = \mu_0 \underline{j}_0$$
$$\underline{j}_0 \times \underline{B}_0 = \underline{\nabla} p_0$$

- after small perturbation from equilibrium, quantities can all be expressed in the form:

$$p = p_0(\underline{x}) + p_1(\underline{x}, t) \qquad \underline{B} = \underline{B}_0(\underline{x}) + \underline{B}_1(\underline{x}, t)$$
$$\underline{u} = \cancel{\underline{u}_0(\underline{x})} + \underline{u}_1(\underline{x}, t) \qquad \text{etc...}$$

$\rightarrow 0$ (no-flow equilib.)

- Since the perturbed system must also satisfy the MHD equations, we ~~use~~ can substitute in the expressions from above, subtract out the equilibrium expressions, keep only first order terms, and incorporate assumptions: (in other words, linearize MHD equations around the equilibrium)

continuity eq: $\frac{\partial}{\partial t}(\rho_0+\rho_1) + \underline{\nabla}\cdot\left[(\rho_0+\rho_1)(\underline{u}_0+\underline{u}_1)\right] = 0$

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho_1}{\partial t} + \underline{\nabla}\cdot(\rho_0\underline{u}_0 + \rho_0\underline{u}_1 + \underbrace{\rho_1\underline{u}_0}_{\substack{0 \\ b/c\ \underline{u}_0=0}} + \underbrace{\cancel{\rho_1\underline{u}_1}}_{\text{second order}}) = 0$$

equilib. continuity eq subtracts out (and, in the no flow case, is zero anyway)

$$\boxed{\frac{\partial \rho_1}{\partial t} + \underline{\nabla}\cdot(\rho_0\underline{u}_1) = 0} \quad (A)$$

equation of motion: $(\rho_0+\rho_1)\left[\left(\frac{\partial}{\partial t} + (\cancel{\underline{u}_0}+\underline{u}_1)\cdot\underline{\nabla}\right)(\cancel{\underline{u}_0}+\underline{u}_1)\right] = (\underline{j}_0+\underline{j}_1)\times(\underline{B}_0+\underline{B}_1) - \underline{\nabla}(p_0+p_1)$

$$(\rho_0+\cancel{\rho_1})\left(\frac{\partial \underline{u}_1}{\partial t} + \cancel{(\underline{u}_1\cdot\underline{\nabla})\underline{u}_1}\right) = \cancel{\underline{j}_0\times\underline{B}_0} + \underline{j}_1\times\underline{B}_0 + \underline{j}_0\times\underline{B}_1 + \cancel{\underline{j}_1\times\underline{B}_1} - \cancel{\underline{\nabla}p_0} - \underline{\nabla}p_1$$

$$\boxed{\rho_0\frac{\partial \underline{u}_1}{\partial t} = \underline{j}_1\times\underline{B}_0 + \underline{j}_0\times\underline{B}_1 - \underline{\nabla}p_1} \quad (B)$$

equation of state: $p\rho^{-\gamma} = \text{constant}$

In this case, do some manipulation before linearizing.

$$\left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)p\rho^{-\gamma} = 0$$

$$\rho^{-\gamma}\left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)p + \underbrace{p\left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)\rho^{-\gamma}}_{-\gamma p\rho^{-\gamma-1}\left(\frac{\partial\rho}{\partial t} + \underline{u}\cdot\underline{\nabla}\rho\right)} = 0$$

$$-\gamma p\rho^{-\gamma-1}\underbrace{\left(\frac{\partial\rho}{\partial t} + \underline{u}\cdot\underline{\nabla}\rho\right)}_{\text{Compare this to continuity eq}}$$

Compare this to continuity eq → simplifies to $\rho\underline{\nabla}\cdot\underline{u}$

$$\cancel{\rho^{-\gamma}}\left(\frac{\partial p}{\partial t} + \underline{u}\cdot\underline{\nabla}p\right) + \gamma p\cancel{\rho^{-\gamma}}\,\underline{\nabla}\cdot\underline{u} = 0$$

Now proceed w/ linearizing around equilibrium ($\underline{u}_0 = 0$):

$$\frac{\partial}{\partial t}(p_0 + p_1) + \underline{u}_1 \cdot \underline{\nabla}(p_0 + \cancel{p_1}) + \gamma(p_0 + \cancel{p_1})\,\underline{\nabla}\cdot\underline{u}_1 = 0$$

equilib. stationary

$$\boxed{\frac{\partial p_1}{\partial t} + \underline{u}_1 \cdot \underline{\nabla} p_0 + \gamma p_0 \underline{\nabla}\cdot\underline{u}_1 = 0} \quad (C)$$

Faraday's Law + induction eq. (curl of Ohm's Law):

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{u} \times \underline{B})$$

$$\frac{\partial}{\partial t}(\underline{B}_0 + \underline{B}_1) = \underline{\nabla} \times \left[(\underline{u}_0 + \underline{u}_1) \times (\underline{B}_0 + \underline{B}_1)\right]$$

$$\frac{\partial \underline{B}_0}{\partial t} + \frac{\partial \underline{B}_1}{\partial t} = \underline{\nabla} \times (\underline{u}_0 \times \underline{B}_0 + \underbrace{\underline{u}_0 \times \underline{B}_1}_{0} + \underline{u}_1 \times \underline{B}_0 + \underbrace{\cancel{\underline{u}_1 \times \underline{B}_1}}_{\text{2nd order}})$$

equilibrium (and zero, anyway)

$$\boxed{\frac{\partial \underline{B}_1}{\partial t} = \underline{\nabla} \times (\underline{u}_1 \times \underline{B}_0)} \quad (D)$$

Ampere's Law: $\underline{\nabla} \times (\underline{B}_0 + \underline{B}_1) = \mu_0 (\underline{j}_0 + \underline{j}_1)$

equilibrium

$$\boxed{\underline{\nabla} \times \underline{B}_1 = \mu_0 \underline{j}_1} \quad (E)$$

Step 2: Let the initial perturbation to the equilibrium be

$$\underline{\xi}(\underline{x}, t) \text{ such that } \frac{\partial \underline{\xi}}{\partial t} = \underline{u}_1$$

$\underline{\xi}$: displacement at time t of the fluid element from position $\underline{x}$

$\underline{u}_1$: velocity perturbation

Initial conditions:

- $\underline{\xi}(\underline{x}, 0) = 0$
- $\underline{u}_1(\underline{x}, 0) \neq 0$ ← "kick" experienced by plasma at time $t=0$ (due, e.g., to random thermal motions)

Therefore:

- $\underline{B}_1(\underline{x}, 0) = 0$; $\rho_1(\underline{x}, 0) = 0$; $p_1(\underline{x}, 0) = 0$ ← other perturbed quantities start from zero at $t=0$
- $\underline{\xi}(\underline{x}, t) = \int_0^t \underline{u}_1(\underline{x}, t')\, dt'$

Rewrite linearized equations of motion using this information:

(A) Integrate: $\int_0^t \frac{\partial \rho_1}{\partial t} dt' = -\int_0^t \underline{\nabla} \cdot (\rho_0 \underline{u}_1) dt'$ ← indep't of time

$$\rho_1(\underline{x}, t) = -\underline{\nabla} \cdot (\rho_0 \underline{\xi})$$

(C) Integrate & rearrange in the same fashion to get:

$$p_1 + \underline{\xi} \cdot \underline{\nabla} p_0 + \gamma p_0 \underline{\nabla} \cdot \underline{\xi} = 0$$

(D) And similarly:

$$\underline{B}_1 = \underline{\nabla} \times (\underline{\xi} \times \underline{B}_0)$$

(E) No integration, but use eq. (D) for $\underline{B}_1$:

$$\underline{j}_1 = \frac{1}{\mu_0} \underline{\nabla} \times (\underline{\nabla} \times (\underline{\xi} \times \underline{B}_0))$$

(B) Finally, incorporate the above expressions for ρ_1 (eq A) p_1 (eq B) and $\underline{j}_1$ (eq E) into the eq. of motion:

$$\rho_0 \frac{\partial^2 \underline{\xi}(\underline{x}, t)}{\partial t^2} = \frac{1}{\mu_0} \underline{\nabla} \times (\underline{\nabla} \times (\underline{\xi} \times \underline{B}_0)) \times \underline{B}_0 + \underline{j}_0 \times (\underline{\nabla} \times (\underline{\xi} \times \underline{B}_0)) + \underline{\nabla}(\underline{\xi} \cdot \underline{\nabla} p_0 + \gamma p_0 \underline{\nabla} \cdot \underline{\xi})$$

$$\boxed{\therefore \rho_0 \frac{\partial^2 \underline{\xi}(\underline{x}, t)}{\partial t^2} = \underline{F}(\underline{\xi}, \text{equilibrium properties})}$$

Note that $\underline{F}$ does not depend on:

- $\underline{u}_1$
- anything besides linear operations on $\underline{\xi}$

(Compare to mechanical system: $m \ddot{\varepsilon} = -\left.\frac{d^2 W}{dx^2}\right|_{x_0} \varepsilon$)

equilib. property ↗ ↖ linear in ε

Step 3:

- It can be shown that the total energy of an ideal MHD system is conserved, so long as the system (which can include plasma & vacuum) is surrounded by a conducting wall. (See description of proof at end of notes.)

This total energy is given by:

$$\mathcal{E} = \int_{V_{total}} \Big(\underbrace{\frac{\rho u^2}{2}}_{\substack{\text{kinetic}\\ \text{energy}\\ \text{density}}} + \underbrace{\frac{p}{\gamma - 1}}_{\substack{\text{thermal}\\ \text{energy}\\ \text{density}}} + \underbrace{\frac{B^2}{2\mu_0}}_{\substack{\text{magnetic}\\ \text{energy}\\ \text{density}}} \Big) d^3x \quad = \text{constant}$$

$$= \underbrace{\int \rho \frac{u^2}{2} d^3x}_{K \text{ (kinetic)}} + \underbrace{\int \left(\frac{p}{\gamma-1} + \frac{B^2}{2\mu_0} \right) d^3x}_{W \text{ (potential)}}$$

In the equilibrium system, $\underline{u}_0 = 0 \longrightarrow K_0 = 0$

$$\mathcal{E}_0 = W_0 = \int \left(\frac{p_0}{\gamma - 1} + \frac{B_0^2}{2\mu_0} \right) d^3x$$

In the perturbed system,

$$K = \int \rho \frac{u_1^2}{2} d^3x \sim \mathcal{O}(\varepsilon^2) \longrightarrow K_1 = 0, \text{ as well}$$

$$K = K_2 + \text{higher order terms}$$

Due to energy conservation,

$$\frac{dW}{dt} = -\frac{dK}{dt} \qquad \left(\frac{d\mathcal{E}}{dt} = 0 \right)$$

$$\frac{d}{dt}\left(W_0 + W_1 + W_2 + \dots \right) = -\frac{d}{dt}\left(\underset{0}{K_0} + \underset{0}{K_1} + K_2 + \dots \right) \quad \text{(from above)}$$

$$\therefore \frac{dW_0}{dt} = 0 \; ; \; \frac{dW_1}{dt} = 0$$

That is, the lowest order term in $\frac{dW}{dt}$ must be the same order as in $\frac{dK}{dt}$.

$$\frac{dW_2}{dt} = -\frac{dK_2}{dt}$$

$$= -\frac{d}{dt}\int \rho_0 \frac{u_1^2}{2}\, d^3x$$

$$= -\int \rho_0 \underbrace{\underline{u}_1}_{\frac{\partial \underline{\xi}}{\partial t}} \cdot \frac{\partial u_1}{\partial t}\, d^3x$$

combine to get $\underline{F}(\underline{\xi})$, as shown in step 2

$$= -\int \underline{\dot{\xi}} \cdot \underline{F}(\underline{\xi})\, d^3x$$

- It can also be shown that $\underline{F}$ is self-adjoint. (again, see end of notes for description of proof)

That is, for any two vector fields $\underline{v}_1$ and $\underline{v}_2$ satisfying the boundary conditions,

$$\int \underline{v}_1 \cdot \underline{F}(\underline{v}_2)\, d^3x = \int \underline{v}_2 \cdot \underline{F}(\underline{v}_1)\, d^3x$$

The self-adjointness of $\underline{F}$ allows us to rewrite $\frac{dW_2}{dt}$ as:

$$\frac{dW_2}{dt} = -\frac{1}{2}\int \left(\underline{\dot{\xi}} \cdot \underline{F}(\underline{\xi}) + \underline{\xi} \cdot \underline{F}(\underline{\dot{\xi}}) \right) d^3x$$

Because $\underline{F}$ depends only on linear operations on $\underline{\xi}$, $\underline{F}\left(\frac{d\underline{\xi}}{dt}\right) = \frac{d}{dt}\underline{F}(\underline{\xi})$

Thus, the integrand is just $\frac{\partial}{\partial t}\left(\underline{\xi} \cdot \underline{F}(\underline{\xi})\right)$.

$$= -\frac{1}{2}\frac{d}{dt}\int \underline{\xi} \cdot \underline{F}(\underline{\xi})\, d^3x$$

$$\boxed{\delta W \equiv W_2 = -\frac{1}{2}\int \underline{\xi} \cdot \underline{F}(\underline{\xi})\, d^3x}$$

(functional)

as previously mentioned, depends on $\underline{\xi}$ (linearly) and equilibrium properties

Compare to 1D mechanical system:

$$\delta W = \frac{\varepsilon^2}{2}\frac{d^2W}{dx^2}\bigg|_{x_0}$$

(function)

Note that, like $\underline{F}$, δW depends only on $\underline{\xi}$, and not $\underline{u}_1$.
(In other words, the path does not matter.)

Conclusion: A necessary & sufficient condition for MHD stability is:

$\delta W(\underline{\xi}) > 0$ for all $\underline{\xi}$ satisfying the boundary conditions of the problem

This is the MHD energy principle.

OK... but what can I actually DO with it?

⇒ Two examples of ways to apply it (from among the future topics listed in the preview section):

1) To find an oscillation frequency ω (if $\delta W > 0$) or growth rate γ (if $\delta W < 0$), look at perturbations that are eigenfunctions of $\frac{-1}{\rho_0(\underline{x})} \underline{F}(\underline{\xi})$.

Mathematically:

$$-\frac{1}{\rho_0(\underline{x})} \underline{F}(\underline{\xi}) = \omega_n^2 \, \underline{\xi}_n(\underline{x})$$

2) Examine more closely the plasma, vacuum, and wall contributions to δW (applying all the appropriate boundary conditions). This leads to an "intuitive" form of the MHD energy principle in which one can identify stabilizing and destabilizing terms.

Bonus notes!

How to prove energy conservation:

In general, to prove local conservation of some quantity, you need to be able to derive something of the form:

$$\frac{\partial}{\partial t}\Big(\underset{\text{quantity}}{\uparrow}\Big) + \nabla\cdot\Big(\underset{\text{flux of that quantity}}{\uparrow}\Big) = 0$$

By manipulating and combining the equations of MHD, you can arrive at just such a relation:

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{\rho u^2}{2} + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}\right)}_{\text{magnetofluid energy density}} + \nabla\cdot\underbrace{\left(\rho\frac{u^2}{2} + \frac{\underline{E}\times\underline{B}}{\mu_0} + \frac{\gamma}{\gamma-1}p\underline{u}\right)}_{\text{energy flux}} = 0$$

The local conservation relation, integrated over the appropriate volumes with the appropriate boundary conditions, can then be used to get a global conservation relation. Different boundary conditions produce different results; for example:

- case 1 = plasma surrounded by perfectly conducting wall
  → $\mathcal{E} = \int\left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}\right)dV$ is conserved in the plasma alone

- case 2 = plasma surrounded by vacuum surrounded by perfectly conducting wall
  → plasma boundary can now move
  → combined plasma-vacuum energy is conserved
  ("Poynting flux" = means by which electromagnetic energy flows between the two)

- case 3 = plasma surrounded by external coils (not perfectly conducting)
  → energy can be supplied to or extracted from system

How to prove self-adjointness of $\underline{F}$

That is, we want to prove that $\int \underline{v}_1 \cdot \underline{F}(\underline{v}_2)\, d^3x = \int \underline{v}_2 \cdot \underline{F}(\underline{v}_1)\, d^3x$

Option A: Expand the integrand and manipulate it until its form is obviously unchanged by $\underline{v}_1 \leftrightarrow \underline{v}_2$ swapping.

One way to do this uses the decomposition:

$$\underline{v}_1 = \underline{v}_{1\perp} + v_{1\parallel}\underline{b}$$
$$\underline{v}_2 = \underline{v}_{2\perp} + v_{2\parallel}\underline{b}$$

components $\perp$ to $\underline{B}_0$ — components $\parallel$ to $\underline{B}_0$

... and a "surprisingly lengthy calculation"...

End result:

$$\int \underline{v}_1 \cdot \underline{F}(\underline{v}_2)\, d^3x = -\int d^3x \Big[\frac{1}{\mu_0} \nabla(\underline{v}_{1\perp} \times \underline{B})_\perp \cdot \underline{\nabla} \times (\underline{v}_{2\perp} \times \underline{B})_\perp$$
$$+ \frac{B^2}{2\mu_0}(\underline{\nabla}\cdot\underline{v}_{1\perp} + 2\underline{v}_{1\perp}\cdot\underline{\kappa})(\underline{\nabla}\cdot\underline{v}_{2\perp} + 2\underline{v}_{2\perp}\cdot\underline{\kappa})$$
$$+ \gamma p(\underline{\nabla}\cdot\underline{v}_1)(\underline{\nabla}\cdot\underline{v}_2)$$
$$- (\underline{v}_{1\perp}\cdot\nabla p)(\underline{v}_{2\perp}\cdot\underline{\kappa}) - (\underline{v}_{2\perp}\cdot\nabla p)(\underline{v}_{1\perp}\cdot\underline{\kappa})$$
$$- \frac{J_\parallel}{2B}(\underline{v}_{1\perp}\times\underline{B})\cdot\underline{\nabla}\times(\underline{v}_{2\perp}\times\underline{B})$$
$$- \frac{J_\parallel}{2B}(\underline{v}_{2\perp}\times\underline{B})\cdot\underline{\nabla}\times(\underline{v}_{1\perp}\times\underline{B})\Big]$$

($\underline{\kappa} = \underline{b}\cdot\underline{\nabla}\,\underline{b}$ = curvature vector)

(reference: Freidberg, Plasma Physics & Fusion Energy, eq 12.17)

Option B: Elegant proof based on:

- conservation of energy
- $\underline{F}$ being independent of $\dot{\underline{\xi}}$
- δK, δW being symmetric functions of their arguments

B/c it is 2nd order: $\delta W = \delta W(\underline{\xi}, \underline{\xi})$ (symmetric function of its arguments)

$\delta \dot{W} = \delta W(\dot{\underline{\xi}}, \underline{\xi}) + \delta W(\underline{\xi}, \dot{\underline{\xi}})$ (take t deriv.)

let $\dot{\underline{\xi}} = \underline{\eta}$ to avoid confusion

$$\frac{1}{2}\,\delta \dot{W} = \delta W(\eta, \xi) = \delta W(\xi, \eta) \quad \left(\begin{array}{c}\text{b/c}\\ \text{symmetric}\end{array}\right)$$

Meanwhile, $\delta K = \int \rho_0 \frac{\dot{\xi}\cdot\dot{\xi}}{2}\, dV = \delta K(\dot{\xi}, \dot{\xi})$ (symmetric funct.)

$$\therefore \quad \delta \dot{K} = \delta K(\ddot{\xi}, \dot{\xi}) + \delta K(\dot{\xi}, \ddot{\xi})$$

Recall that $\ddot{\xi} = \frac{F(\xi)}{\rho_0}$

$$= \delta K\left(\frac{F(\xi)}{\rho_0}, \eta\right) + \delta K\left(\eta, \frac{F(\xi)}{\rho_0}\right)$$

$$= 2\,\delta K\left(\frac{F(\xi)}{\rho_0}, \eta\right) \quad \left(\begin{array}{c}\text{b/c}\\ \text{symmetric}\end{array}\right)$$

From conservation of energy:

$$-\delta \dot{W} = \delta \dot{K}$$

$$-\delta W(\eta, \xi) = \delta K\left(\tfrac{1}{\rho_0} F(\xi), \eta\right) \qquad (i)$$

Because $\dot{\xi} = \eta$ and ξ are <u>independent functions</u>, this must be true for arbitrary η, ξ — for example, if you interchange them:

$$-\delta W(\xi, \eta) = \delta K\left(\tfrac{1}{\rho_0} F(\eta), \xi\right) \qquad (ii)$$

Since the left sides of eq. (i) and (ii) are equal, so are the right sides. Expand these back into integrals (definition of δK) to show self-adjointness of F:

$$\int F(\eta)\cdot \xi \, dV = \int F(\xi)\cdot \eta \, dV$$

reference: Schnack, D.D. "Lecture 22: Proof that the Ideal MHD Force Operator is Self-Adjoint." Lect. Notes Phys. 780, 137–140 (2009)
(Schnack, in turn, attributes I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy. Soc. (London), A244, 17 (1958).)

One last note: If one's initial equilibrium is <u>not</u> stationary ($u_0 \neq 0$), then $F(\xi$, equilibrium, $\dot{\xi})$ is NOT self-adjoint!

explicit dependence on u_0