17.01.2014

Plasma waves: Basics; Unmagnetized waves

Big picture

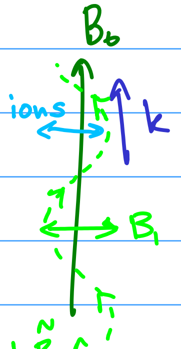
- plasmas can support many types of oscillations, and these can be categorized in several ways:
  - → low frequency (ions oscillating) or high frequency (e^-s oscillating)
  - → unmagnetized ($\underline{B}_0 = 0$) or magnetized ($\underline{B}_0 \neq 0$)
  - → electrostatic ($\underline{B}_1 = 0$) or electromagnetic ($\underline{B}_1 \neq 0$)
  - → "hot" (T dependence) or "cold" (no T dependence)
  - → equations used to solve for them (MHD, 2-fluid, Vlasov)
  - → compressional ($\rho_1 \neq 0$) vs. incompressible ($\rho_1 = 0$)
  - → dispersionless ($\omega(k) \propto k$) vs. dispersive ($\frac{\omega}{k} \neq \frac{\partial \omega}{\partial k}$)

- What does a particular wave "look" like?
  = What quantities are // or ⊥ to one another?

→ Once you've solved for a type of wave, you can draw a diagram of how things are oscillating.

E.g.,

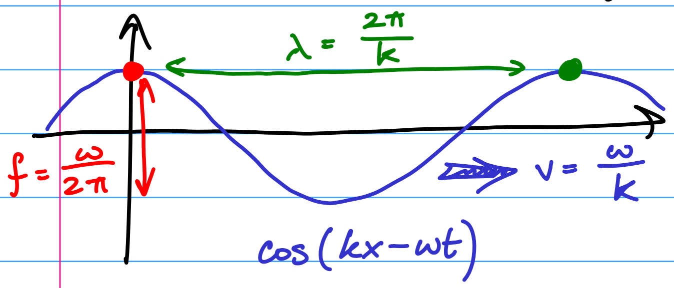
shear Alfven wave (slow wave)



$\underline{B} = B_0 \hat{z} + \tilde{B}_1 \hat{x}$

$\rho_1 = 0$ (no compression)

Aside: Group velocity vs. phase velocity



$\omega = 2\pi f$ (time variation at a point in space)

$k = 2\pi/\lambda$ (spatial variation at a point in time)

The physics of the system determine what waves are possible ⟷ relationship between ω and k.

This relationship provides information about how fast wave packets (wave energy) travel through the system. This may or may not be the same speed at which the individual peaks and troughs travel. (dispersive: matter, deep H_2O; non-disp: light, ideal string)

⇒ Show that wave packets propagate at $\frac{d\omega}{dk}$:

2 waves to make a wave packet: ① $\sin(k_1x - \omega_1 t)$
(assume that both "work" in the medium) ② $\sin(k_2x - \omega_2 t)$

Small differences between the waves:

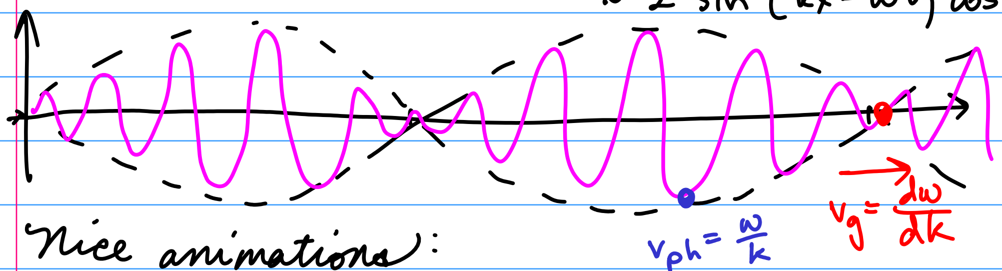
$k_1 = k \qquad k_2 = k + dk$

$\omega_1 = \omega \qquad \omega_2 = \omega + d\omega$

From trig: $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$$\therefore \text{wave ①} + \text{wave ②} = 2\sin\left[\left(k + \frac{dk}{2}\right)x - \left(\omega + \frac{d\omega}{2}\right)t\right]\cos\left(\frac{dk}{2}x - \frac{d\omega}{2}t\right)$$

$$\approx 2\sin(kx - \omega t)\cos\left(\frac{dk}{2}x - \frac{d\omega}{2}t\right)$$



If $\frac{d\omega}{dk} \neq \frac{\omega}{k}$, wave shape changes as it propagates. (Stationary cartoon can't tell you either way.)

Nice animations:

http://physics.usask.ca/~hirose/ep225/animation/dispersion/anim-dispersion.html
http://en.wikipedia.org/wiki/Dispersion_relation

How to find a dispersion relation?

A. Decide what physical system you will be investigating.

1.) What are the fundamental equations describing the medium?
2.) What simplifying assumptions to make (uniform, stationary, importance of terms)?

B. Choose appropriate mathematical tools for the wave phenomena of interest:

1.) small wave amplitude
$\Rightarrow$ linearize equations
2.) plane wave (single frequency, infinite size)
$\Rightarrow$ simplified Fourier analysis (single Fourier mode)

> Aside: Of course, these tools are independent and not always used together, even though we often do so. (E.g., when solving for Landau damping, you still linearize, but then you might use Laplace transformation instead of Fourier transformation.)

C. Do the math and see if you find a relationship between ω and k.

$\Rightarrow$ Possible outcomes:

1.) contradiction → no waves possible, given the assumptions
2.) horrible mess! → try more simplifying assumptions (or more powerful math)
3.) $\omega(k)$ → waves can exist, and now you know their properties (v_{phase}, v_{group})

Nuts and bolts of Part B:

Assume all properties have a form like this ($\underbrace{\text{background}}$ + wave):

may or may not be finite

$$\underline{B} = \underline{B}_0 + \underline{\tilde{B}} \qquad \underline{E} = \underline{E}_0 + \underline{\tilde{E}} \qquad p = p_0 + \tilde{p}$$

e.g., $\underline{\tilde{E}} = \underline{E}_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$

amplitude (constant, small)

$\underline{k}\cdot\underline{r} = k_x x + k_y y + k_z z$

$\underline{k}$ = wave vector

ω = angular frequency

$\frac{\omega}{k}$ = phase velocity (v_{ph})

$\frac{\partial\omega}{\partial k}$ = group velocity (v_g)

Typically*, start by making the following assumptions:

- all background quantities uniform & stationary (not changing in space or time; e.g., $\nabla B_0 = \frac{\partial B_0}{\partial t} = 0$)
- no background flow ($\underline{u}_0 = 0$)
- no background electric field ($\underline{E}_0 = 0$)

$$\underline{B} = \underline{B}_0 + \underline{B}_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)} \qquad p = p_0 + p_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$$

$$\underline{E} = \underline{E}_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)} \qquad \underline{u} = \underline{u}_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$$

*Not surprisingly, some important wave phenomena will occur only in more complex plasmas. But this is a good place to start!

What do the derivatives look like?

- $\frac{\partial}{\partial t}\left(\underline{E}_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}\right) = -i\omega\, \underline{E}_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$

- $\underline{\nabla}\cdot\left(\underline{E}_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}\right) = \frac{\partial}{\partial x}\left(E_{1x} e^{i(\underline{k}\cdot\underline{r} - \omega t)}\right) +$ corresponding terms for y & z

$$= ik_x E_{1x} e^{i(\underline{k}\cdot\underline{r} \cdot \omega t)} + \ldots$$

$$= i\underline{k}\cdot\underline{E}_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$$

- $\underline{\nabla}\times\left(\underline{E}_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}\right) = i\underline{k}\times\underline{E}_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$

- $\underline{\nabla}\left(p_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}\right) = i\underline{k}\, p_1 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$

∴ All differential operators → algebraic operations:

$$\frac{\partial}{\partial t} \longrightarrow -i\omega$$
$$\underline{\nabla} \longrightarrow i\underline{k}$$
$$\underline{\nabla}\cdot \longrightarrow i\underline{k}\cdot$$
$$\underline{\nabla}\times \longrightarrow i\underline{k}\times$$

Note on notation: In the interest of conciseness, $\tilde{\underline{E}} = \underline{E}_1 e^{i(\underline{k}\cdot\underline{r}-\omega t)}$ can be simplified to $\underline{E}_1(\underline{r},t)$ or even $\underline{E}_1$ (now no longer just the constant amplitude), or $\underline{E}$ (when $\underline{E}_0 = 0$). Keep the meaning of terms straight by keeping in mind what assumptions you've made!

Electrostatic vs. electromagnetic waves

As a first exercise, apply what we just proposed

- to Faraday's Law:

$$\underline{\nabla}\times\underline{E} = -\frac{\partial \underline{B}}{\partial t}$$
$$i\underline{k}\times\underline{E}_1 = i\omega\,\underline{B}_1$$

Assumptions: $\underline{E}_0 = 0$
$\underline{B}_0$ stationary
single Fourier mode

- to Gauss's Law:

$$\underline{\nabla}\cdot\underline{E} = \frac{\rho}{\varepsilon_0}$$
$$i\underline{k}\cdot\underline{E}_1 = \frac{\rho_1}{\varepsilon_0}$$

Assumptions: $\underline{E}_0 = 0$ ($\therefore \rho_0 = 0$)
single Fourier mode

Conclude:

① $\underline{k} \parallel \underline{E}_1 \longleftrightarrow \underline{B}_1 = 0$ ← electrostatic

② $\underline{k} \perp \underline{E}_1 \longleftrightarrow \rho_1 = 0$ ← "pure electromagnetic"

(If $\underline{k}$ is neither ∥ nor ⊥ to $\underline{E}_1$, the wave is electromagnetic, but not "pure" electromagnetic.)

A wave that involves no magnetic field oscillations must have a wave vector exactly parallel to the electric field oscillations.
A wave that has no charge density oscillations must have a wave vector exactly perpendicular to the electric field oscillations.

Linearizing (doesn't require assumption of Fourier mode)

Step 1: Substitute assumed expressions for plasma properties into the equations.

E.g., continuity equation:

$$\frac{\partial}{\partial t}(n_0+n_1)+\underline{\nabla}\cdot\left[(n_0+n_1)(\underline{u}_0+\underline{u}_1)\right]=0$$

Step 2: Subtract the background equation: $\frac{\partial n_0}{\partial t}+\underline{\nabla}\cdot(n_0\underline{u}_0)=0$

what's left: $\frac{\partial n_1}{\partial t}+\underline{\nabla}\cdot\left[n_1\underline{u}_0+n_0\underline{u}_1+n_1\underline{u}_1\right]=0$

Step 3: Since the perturbed quantities ($\underline{B}_1$, $\underline{E}_1$, ρ_1, n_1, etc.) are assumed small compared to the unperturbed quantities (e.g., $n_1 \ll n_0$), terms that involve the product of two perturbed quantities can be thrown out.

$$\Rightarrow \boxed{\frac{\partial n_1}{\partial t}+\underline{\nabla}\cdot\left[n_1\underline{u}_0+n_0\underline{u}_1\right]=0}$$

Linearized continuity equation for a given fluid.

Question: We mentioned that some background properties — e.g., $\underline{E}_0$ — will be assumed zero. But then it won't be true that $\underline{E}_1 \ll \underline{E}_0$! Does the argument for linearization still work?

Answer: Yes, so long as you have at least one nonzero background quantity. (Proof involves starting with the property f with the finite background and sufficiently small perturbation $\frac{f_1}{f_0}=\mathcal{O}(\varepsilon)\ll 1$, expressing the other properties as functions of that one, and Taylor expanding.)

Unmagnetized waves ($\underline{B}_0 = 0$)

- plasma waves
- ion acoustic waves
- electromagnetic waves

Plasma waves (a.k.a., plasma oscillations, Langmuir waves, e⁻ plasma wave, Bohm-Gross wave)

Assumptions:

- high-frequency (ions stationary)
  ($n_{i1} = 0$, $\underline{u}_i = 0$)
- electrostatic
  ($\underline{k} \parallel \underline{E} \leftrightarrow \underline{B}_1 = 0$ via Faraday's Law)
- small-amplitude plane wave in uniform bkgd.

Equations to use (2-fluid theory):

① e⁻ eq. of motion:

$$m_e n_e \left(\frac{\partial \underline{u}_e}{\partial t} + (\underline{u}_e \cdot \underline{\nabla}) \underline{u}_e \right) = -e n_e (\underline{E} + \underline{u}_e \times \underbrace{\cancel{\underline{B}}}_{\text{both } \underline{B}_0 \,\&\, \underline{B}_1 = 0}) - \nabla p_e$$

② e⁻ continuity equation:

$$\frac{\partial n_e}{\partial t} + \underline{\nabla} \cdot (n_e \underline{u}_e) = 0$$

(Note: We could also start with the linearized version, but here we'll linearize as we go.)

③ Gauss's Law:

$$\varepsilon_0 \underline{\nabla} \cdot \underline{E} = \rho = e(n_i - n_e)$$

④ equation of state for e⁻s:

$$p_e \cdot n_e^{-\gamma} = \text{const.}$$

Notation: $n_{e1} = n_1$ (only e⁻s move); $\underline{u}_e = \underline{u}_{e1} = \underline{u}_1$ (no background flows, ions not moving)

$n_{i0} = n_{e0} = n_0$

Part A: also assume the plasma is cold ($T_e = 0 \rightarrow p_e = 0$).

Eq ①:

$$m_e \underset{n_0+n_1}{n_e} \left(\overset{-i\omega}{\frac{\partial \overset{\underline{u}_1}{\underline{u}_e}}{\partial t}} + (\underset{\underline{u}_1}{\underline{u}_e} \cdot \overset{i\underline{k}}{\underline{\nabla}}) \underset{\underline{u}_1}{\underline{u}_e} \right) = -e \underset{n_0+n_1}{n_e} \underset{\underline{E}_1}{\underline{E}}$$

becomes

$$m_e (n_0 + \underbrace{n_1)(-i\omega \underline{u}_1}_{} + \underbrace{(\underline{u}_1 \cdot i\underline{k}) \underline{u}_1}_{\text{2nd order}}) = -e(n_0 + \underbrace{n_1) \underline{E}_1}_{}$$

$$+i\omega m_e \cancel{n_0}\,\underline{u}_1 = +e\cancel{n_0}\,\underline{E}_1$$ ← Note that $\underline{u}_1 \parallel \underline{E}_1 \parallel \underline{k}$!

∴ Can drop vector notation.

$$\boxed{i\omega m_e u_1 = eE_1}$$

Eq. ② becomes $$\frac{\partial}{\partial t}(\underset{\sim}{n_0}+n_1) + \underline{\nabla}\cdot\left[(\underset{\sim}{n_0}+n_1)\,\underline{u}_1\right] = 0$$

const. & uniform

2nd order

$$-\cancel{i}\omega n_1 + n_0\cancel{i}\,\underline{k}\cdot\underline{u}_1 = 0$$

parallel

$$\boxed{u_1 = \frac{\omega}{k}\,\frac{n_1}{n_0}}$$

Eq. ③ becomes $$\varepsilon_0\,\underset{i\underline{k}}{\underline{\nabla}}\cdot\underline{E}_1 = e(\cancel{n_0}-\cancel{n_0}-n_1)$$

parallel

$$\varepsilon_0 i k E_1 = -en_1$$

$$\boxed{E_1 = -\frac{en_1}{\varepsilon_0 i k}}$$

∴ We can use equations 2 & 3 to eliminate u_1 & E_1 from eq. 1 and get our relationship between ω and k:

$$① \quad i\omega m_e \underset{\left(\frac{\omega}{k}\frac{n_1}{n_0}\right)}{\underset{②\downarrow}{u_1}} = e\underset{\left(\frac{-en_1}{\varepsilon_0 ik}\right)}{\underset{③\downarrow}{E_1}}$$

$$\frac{i\omega^2 m_e \cancel{n_1}}{\cancel{k}n_0} = -\frac{e^2\cancel{n_1}}{\varepsilon_0 i\cancel{k}}$$ independent of k!

$$\omega^2 = \frac{n_0 e^2}{\varepsilon_0 m_e}$$

$$\boxed{\omega = \sqrt{\frac{n_0 e^2}{\varepsilon_0 m_e}}} \equiv \omega_p = \text{(electron) plasma frequency!}$$

(seen previously)

Part B: This time with finite e^- temperature: $T_{e0} = T_0 > 0$

→ Same as above, except the eq of motion keeps the ∇p_e at the end.

∴ We need to linearize that, too, and then express p_1 in terms of k.

Use eq ④ : $p\, n^{-\gamma} = \text{const.}$ (drop subscripts for now)

$$\underline{\nabla}\cdot(p\, n^{-\gamma}) = 0$$

$$n^{-\gamma}\,\underline{\nabla}\cdot p + p\,\frac{\partial}{\partial n}(n^{-\gamma})\,\underline{\nabla}\cdot n = 0$$ (chain rule)

$$n^{-\gamma}\,\underline{\nabla}\cdot p + -\gamma\, n^{-\gamma-1}\, p\,\underline{\nabla}\cdot n = 0$$

ik

$$\underline{\nabla}\cdot p = \gamma\, n^{-1}\, p\,\underline{\nabla}\cdot n$$

p_0+p_1 n_0+n_1

(where p_0 and n_0 are uniform)

$$i p_1 \underline{k} = \gamma\,\underbrace{\left(\frac{p_0+p_1}{n_0+n_1}\right)}_{\approx \frac{p_0}{n_0} = T_0}\, i n_1 \underline{k}$$ (background is isothermal)

Our linearized eq ① with ∇p_e is:

We know these are ∥, so once again, $\underline{u}_1$ is, too.

from the last line before the boxed version: $-i\omega\, m_e n_0 \underline{u}_1 = -e n_0 \underline{E}_1 - i p_{e1}\underline{k}$

② ③

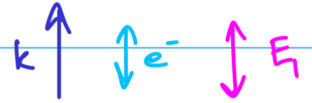
$$-i\omega m_e \cancel{n_0}\left(\frac{\omega}{k}\,\frac{\cancel{n_1}}{\cancel{n_0}}\right) = -e n_0\left(\frac{-e\cancel{n_1}}{\varepsilon_0 i k}\right) - i\gamma \cancel{n_1} T_0 k$$

$$\frac{\omega^2 m_e}{k} = \frac{e^2 n_0}{\varepsilon_0 k} + \gamma T_0 k$$

dispersion relation for plasma waves:

$$\boxed{\omega^2 = \omega_p^2 + \gamma\frac{T_0}{m_e}k^2}$$

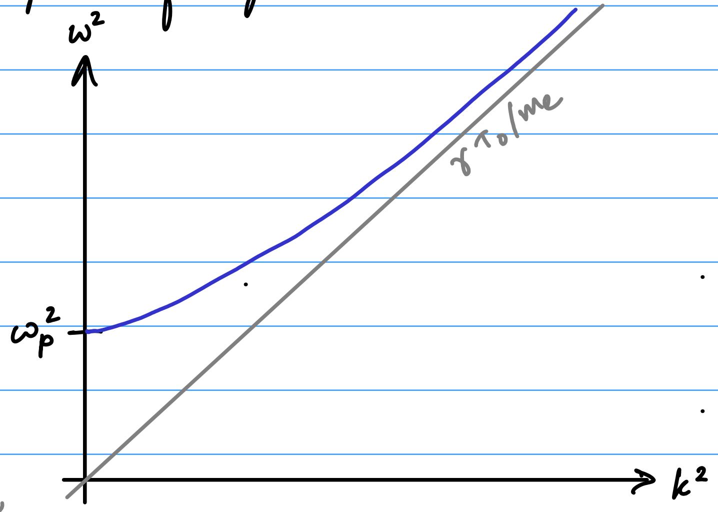
sketch of wave:



(ions stationary)

(Same equations used to find the dispersion relation give information about how the other oscillating quantities (E_1, u_{e1}, n_1, p_{e1}) relate to one another.)

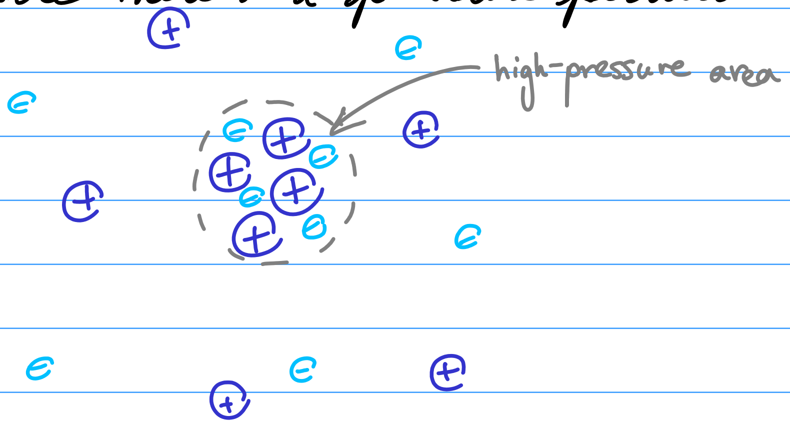
plot of dispersion relation:



Ion acoustic waves (a.k.a., ion sound waves)

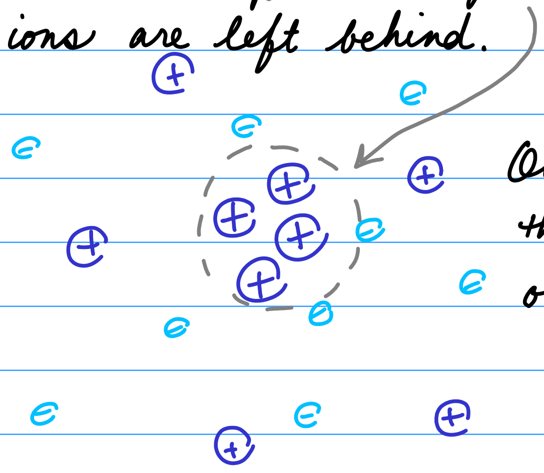
Assumptions:
- low-frequency (ion motion)
- electrostatic ($\underline{k} \parallel \underline{E}_1 \Leftrightarrow \underline{B}_1 = 0$)
- small-amplitude wave in uniform bkgd

Basic physics picture:
- Suppose there is a pressure perturbation:



- Because e^- are more mobile than ions, they spread out from the high-pressure area faster.

- This creates a positive space charge where the ions are left behind.



On a short timescale, this creates plasma oscillations.

- Eventually, on a much longer timescale, the plasma oscillations die out, and the electrons come into equilibrium (force balance between $\underline{E}$ and $\underline{\nabla} p_e$). Note similarity to MHD!

Equations (2-fluid theory):

① e^- eq. of motion (in fluid equilibrium, so there are neither flows nor changes in time):

$$0 = -en_e\underline{E} - \underline{\nabla} p_e$$

$$\underline{E} = \frac{-\underline{\nabla} p_e}{en_e}$$

② ion eq. of motion:

$$m_i n_i \left(\frac{\partial \underline{u}_i}{\partial t} + (\underline{u}_i \cdot \underline{\nabla})\underline{u}_i\right) = qn_i\left(\underline{E} + \cancel{\underline{u}_i \times \underline{B}}\right) - \underline{\nabla} p_i$$

$\underline{B}_0$ and $\underline{B}_1 = 0$

③ ion continuity eq.:

$$\frac{\partial n_i}{\partial t} + \underline{\nabla}\cdot(n_i \underline{u}_i) = 0$$

④,⑤ eq. of state (one for ions, one for e^-s): $p_\alpha n_\alpha^{-\gamma_\alpha} = \text{const.}$
(where $\alpha = e$ or i)

Notation: $n_i = n_0 + n_1$; $n_0 = n_{0e} = n_{0i}$

$\underline{u}_{i1} = \underline{u}_1 = \underline{u}$ (no background flows, e^- in equilibrium)

Use e^- force balance (eq. 1) to eliminate $\underline{E}$ from the ion eq. of motion (eq. 2):

$$\text{Eq. ②: } m_i n_i \left(\frac{\partial \underline{u}_i}{\partial t} + (\underline{u}_i\cdot\underline{\nabla})\underline{u}_i\right) = q n_i \overbrace{\left(\frac{-\underline{\nabla} p_e}{e n_e}\right)}^{\text{from ①}} - \underline{\nabla} p_i$$

$$= -\underbrace{\left(\frac{q n_i}{e n_e}\right)}_{\sim 1 \text{ due to quasineutrality}} \underline{\nabla} p_e - \underline{\nabla} p_i$$

$$\underset{\underset{\text{const.}}{n_0}+n_1}{\underset{\uparrow}{m_i n_i}}\left(\overset{-i\omega}{\frac{\partial \underline{u}_i}{\partial t}} + \underbrace{(\underline{u}_i\cdot\overset{i\underline{k}}{\underline{\nabla}})\underline{u}_i}_{u_1}\right) = -\underline{\nabla} p_e - \underline{\nabla} p_i$$

(→ term is 2nd order)

linearize as before ($i\gamma T_o n_1 \underline{k}$ for each)

$$-\cancel{i}\omega m_i \underbrace{(n_0+n_1)\underline{u}_1}_{\text{2nd order}} = -\cancel{i}\underline{k}\left(\gamma_e T_{eo} n_{e1} + \gamma_i T_{io} n_1\right)$$

assume quasi-neutrality on scale of perturbation

$$n_{e1} = Z n_{i1} = Z n_1$$

$$\boxed{\omega m_i n_0 \underline{u}_1 = \underline{k} n_1 (\gamma_e Z T_{eo} + \gamma_i T_{io})} \quad \therefore\ \underline{k} \parallel \underline{u}_1$$

$$\text{Eq. ③, linearized: } \overset{\displaystyle\frac{\partial}{\partial t}}{} \underset{-i\omega}{\frac{\partial}{\partial t}}(\underset{\text{const.}}{n_0}+n_1) + \underset{i\underline{k}}{\underline{\nabla}}\cdot\left[\underbrace{(n_0+n_1)\underline{u}_1}_{\text{2nd order}}\right] = 0$$

$$-\cancel{i}\omega n_1 + \cancel{i} n_0 \underbrace{\underline{k}\cdot\underline{u}_1}_{\text{parallel}} = 0$$

$$\boxed{u_1 = \frac{\omega}{k}\frac{n_1}{n_0}}$$

Combine the two boxed equations:

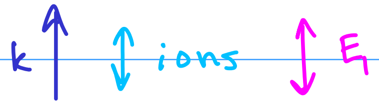
$$\omega^2 \frac{m_i \cancel{n_0} \cancel{n_1}}{k \cancel{n_0}} = k \cancel{n_1}\left(\gamma_e Z T_{eo} + \gamma_i T_{io}\right)$$

dispersion relation for ion acoustic waves:

$$\boxed{\omega = k\sqrt{\frac{\gamma_e Z T_e + \gamma_i T_i}{m_i}}}$$

non-dispersive $\left(\frac{\omega}{k} = \frac{d\omega}{dk}\right)$

sketch of wave:



(e^-s in force balance)

Discussion:

- When $T_e \approx T_i$, the phase velocity (and group velocity) is close to the ion thermal speed ($\sqrt{\frac{T_i}{M_i}}$); as a result, the waves are heavily damped.
- Therefore, ion acoustic waves are most often observed in plasmas with $T_e >> T_i$.
- Assuming $T_e >> T_i$, electrons are isothermal ($\gamma_e = 1$), and $z = 1$ (hydrogen), the wave velocity reduces to:

$$\frac{\omega}{k} \equiv \boxed{c_s = \sqrt{\frac{T_e}{M_e}}}$$

This reflects that the wave arises from "competition" between

← e^- pressure and

← ion mass.

ion sound speed

Electromagnetic waves (light waves)

Because the plasma responds to the EM fields of the wave, $j_1 \neq 0$ (unlike EM wave propagation in a vacuum).

assumptions:
- high frequency (e^- motion)
- pure electromagnetic

  ($\underline{k} \perp \underline{E} \leftrightarrow \rho_1 = 0$) ← From Gauss's Law. (We solved for this before, but this is our first time using it for a wave.)
- small-amplitude in uniform bkgd. (ions stationary, no bkgd flow)

Notation: $\underline{u}_e = \underline{u}_1$; $n_{e1} = n_1$; $\rho_1 = \rho_{e1}$

$n_{e0} = n_{i0} = n_0$

Equations (from 2-fluid theory):

① e^- eq. of motion:

$$m_e n_e \left(\frac{\partial \underline{u}_e}{\partial t} + (\underline{u}_e \cdot \underline{\nabla})\, \underline{u}_e \right) = -e n_e \left(\underline{E} + \underbrace{\underline{u}_e \times \underline{B}}_{\uparrow} \right) - \underline{\nabla} p_e$$

Even though we have $\underline{B}_1$ this time, this term still goes away when we linearize, because it is 2nd order ($\underline{u}_1 \times \underline{B}_1$).

linearized:

$$-i\omega\, m_e n_0 \underline{u}_1 = -e n_0 \underline{E}_1 - i p_1 \underline{k}$$

② e^- continuity eq (n_0 uniform, $\underline{u}_0 = 0$, linearized):

$$\omega n_1 = n_0\, \underline{k} \cdot \underline{u}_1$$

③ e^- eq of state:

$$p n^{-\gamma} = \text{const.} \xrightarrow{\text{linearized}} p_1 = i\gamma T_0 n_1$$

④ Faraday's Law (linearized):

$$\underline{k} \times \underline{E}_1 = \omega \underline{B}_1$$

⑤ Ampere's Law (w/ Maxwell's correction):

$$\underline{\nabla} \times \underline{B} = \mu_0 j + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

linearized: $i\underline{k} \times \underline{B}_1 = \mu_0 j_1 - \frac{i\omega}{c^2} \underline{E}_1$

Since ions are stationary, current is due to e^- motion only:

$$j = \sum_\alpha q_\alpha n_\alpha \underline{u}_\alpha = -e n_e \underline{u}_e$$

linearized: $j_1 = -e n_0 \underline{u}_1$ ($\underline{u}_0 = 0$ and $n_1 \underline{u}_1$ is 2nd order)

Substituting this in, eq. ⑤ becomes:

$$i\underline{k} \times \underline{B}_1 = -e\mu_0 n_0 \underline{u}_1 - \frac{i\omega}{c^2} \underline{E}_1$$

This tells us that $\underline{u}_1 \parallel \underline{E}_1 \perp \underline{k}$, because:

$$\underline{k}\cdot\Big(\underbrace{i\underline{k}\times\underline{B}_1}_{\perp\underline{k}} = -e\mu_0 n_0 \underset{\substack{\uparrow\\ \therefore\ \underline{k}\cdot\underline{u}_1 = 0}}{\underline{u}_1} - \frac{i\omega}{c^2}\underset{\substack{\uparrow\\ \perp\underline{k}}}{\underline{E}_1}\Big)$$

That $\underline{u}_1 \perp \underline{k}$ tells us:

- $n_1 = 0$ (from eq. 2)
- $p_1 = 0$ (from eq. 3, since $n_1 = 0$)

Eq ① then simplifies to:

$$i\omega m_e \underline{u}_1 = e\underline{E}_1$$

Substitute eq ① and ④ into ⑤:

$$i\underline{k}\times\underline{B}_1 = -e\mu_0 n_0\underline{u}_1 - \frac{i\omega}{c^2}\underline{E}_1$$

$$\downarrow ④ \qquad\qquad ① \searrow$$

$$i\underline{k}\times\left(\frac{\underline{k}\times\underline{E}_1}{\omega}\right) = -e\mu_0 n_0\left(\frac{e\underline{E}_1}{i\omega m_e}\right) - \frac{i\omega}{c^2}\underline{E}_1$$

$$\underbrace{\underline{k}\times(\underline{k}\times\underline{E}_1)}_{\substack{-k^2\underline{E}_1\\ (\text{b/c } \underline{k}\perp\underline{E}_1)}} = \left(\underset{\substack{\searrow\\ \mu_0\varepsilon_0\left(\frac{e^2 n_0}{m_e\varepsilon_0}\right)}}{\frac{\mu_0 e^2 n_0}{m_e}} - \frac{\omega^2}{c^2}\right)\underline{E}_1$$

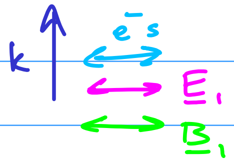
$$k^2 = \frac{1}{c^2}\left(\omega_p^2 - \omega^2\right)$$

dispersion relation for electromagnetic waves (in an unmagnetized plasma)

$$\boxed{\omega^2 = c^2k^2 + \omega_p^2}$$

As $n_0 \to 0$, $\omega^2 \to c^2k^2$ (vacuum EM waves).

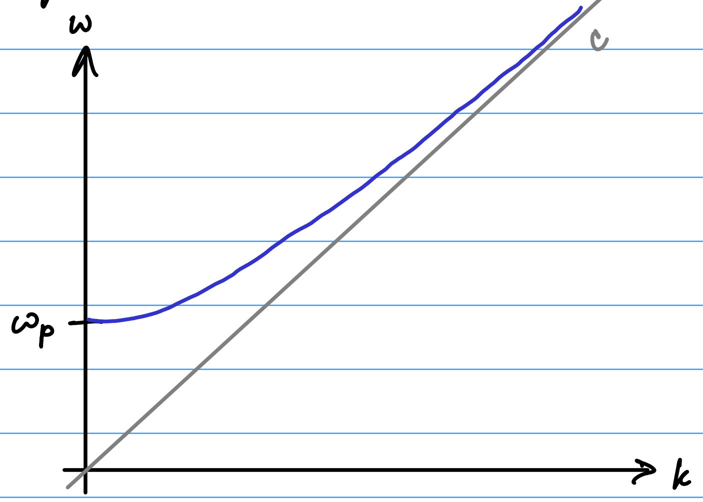
sketch of wave:



($\underline{u}_1$, $\underline{E}_1$, and $\underline{B}_1$ are not all ∥ or in phase, of course; just all ⊥ to $\underline{k}$)

$p_1 = 0$; $n_1 = 0$

plot of dispersion relation:



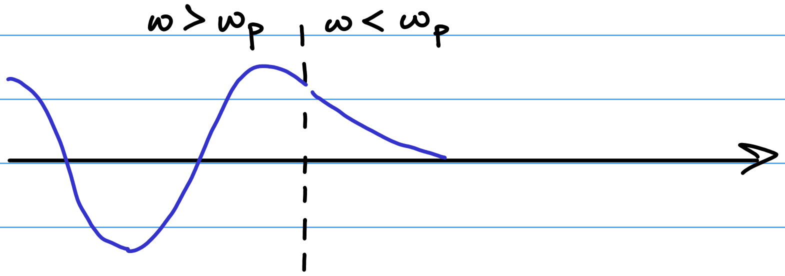
Frequencies $\omega < \omega_p$ cannot propagate, since $k^2 < 0$.

$$\text{Let } k^2 = -\kappa^2 \quad \text{where } \kappa > 0$$

$$k = \pm i\kappa$$

$$e^{i(\underline{k}\cdot r - \omega t)} = \underbrace{e^{\pm\kappa\cdot\underline{r}}}_{\text{spatial exponential decay}} \underbrace{e^{-i\omega t}}_{\text{(still oscillating in time)}}$$

Imagine a medium with a very gentle variation in density. Beyond the point where $\omega < \omega_p$, waves become evanescent:



∴ Use EM waves to measure plasma density:

① Scan the frequency of your antenna until your receiver on the other side of the plasma detects a signal.

② Look at the change in phase delay due to the presence of the plasma (as compared to vacuum).